CNCM Online Round 1

CNCM Administration

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Problems.

Problem 1. Pooki Sooki has 8 hoodies, and he may wear any of them throughout a 7 day week. He changes his hoodie exactly 2 times during the week, and will only do so at one of the 6 midnights. Once he changes out of a hoodie, he never wears it for the rest of the week. The number of ways he can wear his hoodies throughout the week can be expressed as $\frac{8!}{2^k}$. Find k.

Problem 2. Akshar is reading a 500 page book, with odd numbered pages on the left, and even numbered pages on the right. Multiple times in the book, the sum of the digits of the two opened pages are 18. Find the sum of the page numbers of the last time this occurs.

Problem 3. Define $S(N)$ to be the sum of the digits of N when it is written in base 10, and take $S^k(N) = S(S(\dots(N)\dots))$ with k applications of S. The *stability* of a number N is defined to be the smallest positive integer K where $S^K(N) = S^{K+1}(N) = S^{K+2}(N) = \dots$. Let T_3 be the set of all natural numbers with stability 3. Compute the sum of the two least entries of T_3.

Problem 4. Consider all possible pairs of positive integers (a, b) such that $a \ge b$ and both $\dfrac{a^2+b}{a-1}$ and $\dfrac{b^2+a}{b-1}$ are integers. Find the sum of all possible values of the product ab.

Problem 5. Positive reals $a, b, c \le 1$ satisfy $\frac{a+b+c-abc}{1-ab-bc-ca} = 1$. Find the minimum value of

$$\left(\frac{a+b}{1-ab} + \frac{b+c}{1-bc} + \frac{c+a}{1-ca}\right)^2.$$

Problem 6. In triangle $\triangle ABC$ with $BC = 1$, the internal angle bisector of $\angle A$ intersects BC at D. M is taken to be the midpoint of BC. Point E is chosen on the boundary of $\triangle ABC$ such that ME bisects its perimeter. The circumcircle ω of $\triangle DEC$ is taken, and the second intersection of AD and ω is K, as well as the second intersection of ME and ω being L. If B lies on line KL and ED is parallel to AB, then the perimeter of $\triangle ABC$ can be written as a real number S. Compute $\lfloor 1000S \rfloor$.

Problem 7. Three cats–TheInnocentKitten, TheNeutralKitten, and TheGuiltyKitten labelled P_1, P_2, and P_3 respectively with $P_{n+3} = P_n$–are playing a game with three rounds as follows:

1. Each round has three turns. For round $r \in \{1, 2, 3\}$ and turn $t \in \{1, 2, 3\}$ in that round, player P_{t+1-r} picks a non-negative integer. The turns in each round occur in increasing order of t, and the rounds occur in increasing order of r.
2. **Motivations:** Every player focuses primarily on maximizing the sum of their own choices and secondarily on minimizing the total of the other players' sums. TheNeutralKitten and TheGuiltyKitten have the additional tertiary priority of minimizing TheInnocentKitten's sum.
3. For round 2, player P_2 has no choice but to pick the number equal to what player P_1 chose in round 1. Likewise, for round 3, player P_3 must pick the number equal to what player P_2 chose in round 2.
4. If not all three players choose their numbers such that the values they chose in rounds 1,2,3 form an arithmetic progression in that order by the end of the game, all players' sums are set to -1 regardless of what they have chosen.
5. If the sum of the choices in any given round is greater than 100, all choices that round are set to 0 at the end of that round. That is, rules 2, 3, and 4 act as if each player chose 0 that round.
6. All players play optimally as per their motivations. Furthermore, all players know that all other players will play optimally (and so on.)

Let A and B be TheInnocentKitten's sum and TheGuiltyKitten's sum respectively. Compute $1000A + B$ when all players play optimally.

§1 Solutions.

Solution 1. First, we know that there are $8 * 7 * 6$ ways to choose 3 hoodies to wear, and the order of when to wear them. We multiply this to the number of ways he picks days to change his hoodie. There are 6 chances for him to change his hoodie, and he will do so exactly 2 times. So, there are $\binom{6}{2} = 15 = 5 * 3$ ways for him to do so. Multiplying these two numbers together, we get $8 * 7 * 6 * 5 * 3$ which is the same as $8!/(4 * 2)$ so our answer is $\boxed{3}$.

Solution 2. Let $\underline{(a)(b)(c)}$ denote the 3-digit number with digits a, b, and c.

First, we prove that the sum of the digits on the 2 pages must be odd when the 2 numbers have the same tens digit. This is because the same hundreds and tens place sum to an even number, while the two consecutive ones digits sum to an odd number, and even + odd = odd.

Next, we prove that the sum of the digits must be odd when the 2 numbers have different hundreds digits. This is only possible when the two numbers are $\underline{(z)(9)(9)}$ $\underline{(z+1)(0)(0)}$ for some integer z. The hundreds digits sum to odd, the tens digit sums to odd, and the ones digit sums to odd, so odd + odd + odd = odd.

So now, we know that the 2 pages must have the same hundreds place, and different tens place. So we set our two variables x, y as $\underline{(x)(y)(9)}$ and $\underline{(x)(y+1)(0)}$. By taking the sum of the digits, we get $x + y + 9 + x + (y + 1) + 0 = 18$ which after simplifications becomes $x + y = 4$. Since we would like the last instance in the book that the digits sum to 18, we take $x = 4$ and quickly verify that $409, 410$ works, so our answer is $\boxed{819}$.

Solution 3. Clearly, the two least numbers are less than 1000, which can be motivated by just playing around with the problem. Assume $N < 1000$. Now, We bound the following: $S^1(N) \leq 27, S^2(N) \leq 10$, since the maximal sum of digits of a number less than 1000 occurs at 999, and the maximal sum of digits of a number less than $9 + 9 + 9 = 27$ occurs at 19, which gives a sum of 10.

At this point, if $a_2(N)$ is not 10, then the stability of N is just 2, since all 1-digit numbers are stable. Hence, take $a_2(N) = 10 \implies a_1(N) \geq 19$ is the minimal choice. 199 is the smallest positive integer with a digit sum not less than 19, and is therefore the smallest element of T_3, and from here we get 289 as the next number. Adding gives $199 + 289 = \boxed{488}$.

Solution 4. (**Note:** The notation $a|b$ denotes that a divides into b.)

The conditions can be rewritten as $a - 1|a^2 + b$ and $b - 1|b^2 + a$.

Note that $a-1|a^2-a$. We can thus rewrite the first condition as $a-1|(a^2-a)+(a+b) \implies a-1|a+b$. Also note that $a-1|a-1$, so we can rewrite the previous statement as $a-1|(a-1)+b+1 \implies a-1|b+1$.

Repeating this for the second condition, we get $b - 1|a + 1$.

From $a - 1|b + 1$, we get that $b + 1 \geq a - 1 \implies a \leq b + 2$. From $b - 1|a + 1$, we get that $a + 1 \geq b - 1 \implies a \geq b - 2$. From these two inequalities, we get that $b - 2 \leq a \leq b + 2$. Since the problem states that $a \geq b$, we can restrict this to $b \leq a \leq b + 2$. This gives us three cases, $a = b, a = b + 1, a = b + 2$ that we can use on our conditions $a - 1|b + 1$, $b - 1|a + 1$.

Case 1: $a = b$ The two conditions can both be written as $b - 1|b + 1 \implies b - 1|2$. This gives us the solutions $b = 2, 3$, which return the ordered pairs $(2, 2), (3, 3)$.

Case 2: $a = b + 1$ The conditions can be written as $b|b + 1$ and $b - 1|b$. This is clearly never true, so this returns no ordered pairs.

Case 3: $a = b + 2$ The conditions can be written as $b + 1|b + 1$ and $b - 1|b + 3 \implies b - 1|4$. The first condition is clearly always true. The second condition gives us the solutions $b = 2, 3, 5$, which return the ordered pairs $(4, 2), (5, 3), (7, 5)$.

Our solutions for (a, b) are $(2, 2), (3, 3), (4, 2), (5, 3), (7, 5)$. Taking the products ab and summing them up, we get $4 + 9 + 8 + 15 + 35 = \boxed{71}$.

Solution 5. Note that each term look a lot like the tangent addition formula. Thus, let $a = \tan(x), b = \tan(y), c = \tan(z)$ to obtain

$$\sum_{\text{cyc}} \frac{a+b}{1-ab} = \sum_{\text{cyc}} \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} = \sum_{\text{cyc}} \tan(x+y)$$

Since $\tan(x)$ is convex on the interval $(0, \frac{\pi}{2})$, we may use Jensen's inequality to get that

$$\sum_{\text{cyc}} \tan(x+y) \geq 3\tan\left(\frac{2(x+y+z)}{3}\right)$$

If we note that the condition is equivalent to $x + y + z = \frac{\pi}{4}$ due to the tangent addition formula (we are motivated by the massive simplification earlier), we find that the minimum value is equal to $(3\tan(\frac{\pi}{6}))^2 = (\sqrt{3})^2 = \boxed{3}$ which is achievable at $x = y = z = \frac{\pi}{12}$ or $a = b = c = 2 - \sqrt{3}$.

Solution 6. It is trivial to observe the following few relations.

1. $EM \parallel AD$
2. $KL \parallel AC$
3. $AC = 1$, which follows by similarity $\triangle ABC \sim \triangle EDC$ and Angle Bisector Theorem.

We now begin to length chase; set $AB = a, AC = b$. Consider a real $r > 0$ and $BD = ra, DC = rb$ by Angle Bisector Theorem. We prove the following:

Claim 1.1 — $(E, L; C, K)$ is harmonic.

Proof. Consider $(M, P_\infty; C, B)$. We know

$$(M, P_\infty; C, B) \equiv_L (E, L; C, K) = -1$$

and the claim follows. □

Hence, we may conclude that CK is the C-symmedian of $\triangle ECL$. In fact, we know that K, D are isogonal conjugates with respect to $\angle ECL$, and thus M is the midpoint of EL. By homothety and observation 1, D is the midpoint of $AD = DN$. Hence, Power of a Point on D yields the following:

$$AD = DN = r\sqrt{ab}.$$

Stewart's on $\triangle ABC$ with respect to AD yields $a = \sqrt{2} - 1$. In fact, the final answer must be

$$\lfloor 1000S \rfloor = \lfloor 1000(\sqrt{2}+1) \rfloor = \boxed{2414}.$$

Solution 7. Before we start this long and arduous proof, please note that this is not how this problem was intended to be solved during the contest That is, this proof is focused on rigor, not speed or efficiency. The actual intended solution was to guess and check to observe a pattern, and then intuit the answer using either algebra or logic. Some of this logic will be explored during the proof, and some of the intuition will be discussed at the very end.

Firstly, denote the P_n's choice in round r as $C_{(n,r)}$. In this proof, any reference to a rule will be explicitly stated at "Rule N". Specifically for Rule 2, the primary, secondary, and tertiary priorities will be denoted 2a, 2b, and 2c respectively (this rule will also be used implicitly many times as well). Let's proceed by demonstrating the optimal game where each entry represents the choice with the row representing the round and column representing the player.

$$\begin{bmatrix} 1 & 1 & 97 \\ 49 & 1 & 49 \\ 97 & 1 & 1 \end{bmatrix} \tag{1}$$

We will demonstrate why this is optimal through a series of observations and lemmas.

Lemma 1.2

If Rule 5 is violated in round 1, all players' sums will be 0. If Rule 5 is violated in round 2, all players' sums will be -1. Note that any instance where all players choose 0 initially counts as a violation in this lemma.

We prove the first part first. If Rule 5 is violated in round 1, it follows that $C_{(3,2)} = 0$ or else Rule 4 would be violated. However, P_3 would actually choose some arbitrarily large number so that $C_{(1,2)} = 0$ as well so as to follow Rule 2b. The rest follows.
As for the second part, it is clear that Rule 4 would be violated for at least one player. We now apply this lemma to set a bound.

Lemma 1.3

Very simply, we can bound $C_{(1,1)} \leq 50$.

Assume that $x = C_{(1,1)} > 50$ and that Rule 5 is not violated in round 3. Further assume that Rule 5 is not violated at all. From this, we get that obtain the following matrix depicting the minimum sums:

$$\begin{bmatrix} x & y & 0 \\ \frac{x}{2} & x & \frac{x}{2} \\ 0 & 2x-y & x \end{bmatrix} \tag{2}$$

This implies that the minimum total sum is greater than 300, which further implies that the sum in at least one of the rounds is greater than 100, violating Rule 4. By Lemma 1.2, this shows that it is more optimal for the players to violate Rule 5 in round 1, showing that the maximal sums are 0.

Now we prove the final result. There is indeed a way for the players to get a positive score. That situation is demonstrated here:

$$\begin{bmatrix} x & 2x & y \\ \frac{x}{2} & x & \frac{y}{2} \\ 0 & 0 & 0 \end{bmatrix} \tag{3}$$

Note that since $C_{(3,3)} \neq C_{(2,2)}$, Rule 5 must have been violated in round 3. Taking a look at round 1, we see that $x \leq 33$ in order for Rule 4 to not be violated. This implies that P_1's maximal sum is 48 (notice parity) in this situation. Since we know that P_1 can achieve a sum of 147 from (1), we see that $C_{(1,1)} \leq 50$ as desired.

Lemma 1.4

$C_{(1,1)} \neq 0$.

It helps to take P_2's perspective. Since P_2's maximal sum is 0, he would in accordance with Rule 2b purposefully violate Rule 5 in round 1. Thus by Lemma 1.2, all players' sums would be 0. This is obviously nonoptimal.

Lemma 1.5

The total of P_1 and P_2's sums does not change as long as $C_{(1,1)}$ is held constant and the parity of $C_{(2,1)}$ is constant.

Note that when $C_{(1,1)}$ is constant, P_2's sum is constant due to Rule 4. Also note that Rule 5 will not be violated due to Lemma 1.2. Thus it suffices to prove that the total of all players' sums is equal to 300 and 297 for even and odd $C_{2,1}$ respectively. Before we prove this, we obtain some inequalities from the following:

$$\begin{bmatrix} x & y & ? \\ ? & x & ? \\ ? & 2x-y & x \end{bmatrix} \tag{4}$$

$$x + y \leq 100 \tag{5}$$

$$3x - y \leq 100 \tag{6}$$

Now we prove the former result:

$$\begin{bmatrix} x & y & 100-x-y \\ 50-x+\frac{y}{2} & x & 50-\frac{y}{2} \\ 100-3x+y & 2x-y & x \end{bmatrix} \tag{7}$$

We determine that this works always. For this to not be nonoptimal, we cannot violate Rules 4, 5, or 1. Thus, $100 - x - y \geq 0$ and $100 - 3x + y \geq 0$. These are proved by (5) and (6) respectively. We also see that $C_{(3,2)}$ and $C_{(1,2)}$ are integers as y is even. Finally, we can determine that this is optimal for P_1 and P_3 as it is obvious that reducing any of their picks (other than x) will reduce their sum, violating Rule 2a. Now we tackle the latter case.

$$\begin{bmatrix} x & y & 99-x-y \\ \frac{99-2x+y}{2} & x & \frac{99-y}{2} \\ 99-3x+y & 2x-y & x \end{bmatrix} \tag{8}$$

Unfortunately, we have to use a bit more effort to prove this as the inequalities do not directly work. First, we prove that $99 - x - y \geq 0$. For sake of contradiction, assume that this is false. Then from (5), we get that $x + y = 100$. This implies that x is odd. However, we know that $C_{(3,1)} \equiv C_{(3,3)} \mod 2$. Thus we have hit a contradiction. Similar reasoning with $C_{(1,3)}$ proves that that works as well.

Note that the main difference is that $C_{(3,2)}$ and $C_{(1,2)}$ are decreased by $\frac{1}{2}$ due to parity. They cannot be increased as that would violate previously mentioned inequalities. Thus, we may conclude this is optimal, completing the proof of this lemma. We can derive from this an immediate corollary.

Lemma 1.6

$C_{(2,1)}$ is always odd. and $C_{(1,1)} \neq 50$.

It is obvious by Lemma 1.5 (note that the total of P_1 and P_2's sums is $297 - 3C_{(1,1)}$ for the odd case) and Rule 2b that $C_{(2,1)}$ will be odd if possible. We prove that this is always possible. First, we eliminate $C_{(1,1)} = 50$.

$$\begin{bmatrix} 50 & 50 & 0 \\ 25 & 50 & 25 \\ 0 & 50 & 50 \end{bmatrix} \tag{9}$$

Since this the only possible configuration that does not violate Rule 4 or 5, this is the only configuration that will result. Obviously, this results in a worse sum for P_1, so $C_{(1,1)} \neq 50$. Now we claim that $C_{(2,1)} = C_{(1,1)}$ and $C_{(2,1)} = C_{(1,1)} + 1$ work for when $C_{(1,1)}$ is odd and even respectively. Using (8), we see that it fairly obviously works so we are done.

Lemma 1.7

$C_{(2,1)}$ will always be least odd number such that Rule 4 and 5 are not violated.

Due to Lemma 1.5 and Lemma 1.6, P_2 can ignore Rule 2a and 2b. That is, P_2's goal is now to simply minimize P_1's sum. From Rule 4 we see that each player's sum is equal to triple their choice in round 2. Combining these statements with (8) proves this lemma.

Lemma 1.8

$C_{(2,1)} = 1$ when $C_{(2,1)} \leq 33$. Otherwise, if $x = C_{(1,1)}$ is odd, $C_{(2,1)} = 3x - 98$, and if x is even, $C_{(2,1)} = 3x - 99$.

Due to Lemma 1.7, we only need to calculate the lower bounds. So, we will primarily be using the inequality derived in Lemma 1.5 that $y = C_{(2,1)} \geq 3x - 99$ along with the fact that $y \geq 0$. In fact, these trivialize the proof of this lemma.

Putting It Together. Due to Lemma 1.8, we present three configurations for when $C_{(2,1)} \leq 33$, $C_{(2,1)} = 3x - 98$, and $C_{(2,1)} = 3x - 99$:

$$\begin{bmatrix} x & 1 & 98 - x \\ 50 - x & x & 49 \\ 100 - 3x & 2x - 1 & x \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} x & 3x - 98 & 197 - 4x \\ \frac{x+1}{2} & x & \frac{197-3x}{2} \\ 1 & 98 - x & x \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} x & 3x - 99 & 198 - 4x \\ \frac{x}{2} & x & 99 - \frac{3x}{2} \\ 0 & 99 - x & x \end{bmatrix} \tag{12}$$

For the first configuration, minimizing x would maximize P_1's sum. Thus, $x = 1$ would produce a sum of 147. For the second configuration, $x = 49$ would produce a sum of 75. Lastly for the third configuration, $x = 48$ would produce a sum of 72. So thus we may conclude that $C_{(1,1)} = 1$ by Rule 2a. Here's the final configuration:

$$\begin{bmatrix} 1 & 1 & 97 \\ 49 & 1 & 49 \\ 97 & 1 & 1 \end{bmatrix} \tag{13}$$

Our final answer then is $147 \cdot 1000 + 147 = \boxed{147147}$

Intuition. Read this at least after reading through Lemma 1.5 (maybe even just 1.4). This section is here to as a reference for why minimizing $C_{(1,1)}$ would be optimal. Firstly, we intuitively prove Lemma 1.6. Very simply, we need to make sure that the round 2 choices are all integers and not decimals. This means the round 1 and round 3 choices must have equal parity. If $C_{(2,1)}$ is odd, then the first round sum is reduced from 100 as the other round 1 choice have equal parity.

Now we cover Lemma 1.7. Since P_2's sum is constant, he wants to minimize P_1's sum. By reducing $C_{(2,1)}$, P_3 can choose greater numbers in rounds 1 and 2. This reduces $C_{(1,2)}$, which reduces P_1's sum as desired.

Finally we finish the intuitive proof. When $C_{(1,1)}$ is changed, $C_(3,3)$ changes by an equal amount whereas $C_{(3,1)}$ changes by an opposite amount (this is since $C_{(2,1)}$ is constant). This implies $C_{(3,2)}$ is constant so P_3's sum is constant. This means P_1 would only need to reduce P_2's sum as that would mean his own sum increases. This is done by simply minimizing $C_{(1,1)}$, and so we get that it is equal to 1. Then we only need to check the above configuration to get the answer of $\boxed{147147}$.